Marcello Chiodi: GENERATION OF PSEUDO RANDOM VARIATES FROM A NORMAL DISTRIBUTION OF ORDER P.Pubblicato su Statistica Applicata (Italian Journal of Applied Statistics) 1995, Vol.7, n.4, pagg. 401-416

GENERATION OF PSEUDO RANDOM VARIATES FROM A NORMAL DISTRIBUTION OF ORDER P.

Marcello Chiodi

Istituto di Statistica,
Facoltà di Economia e Commercio di Palermo.

Riassunto

Negli studi di simulazione riguardanti il comportamento di particolari stimatori sotto l'allontanamento dalle ipotesi di normalità, la famiglia delle curve normali di ordine p svolge un ruolo fondamentale, dal momento che al variare del parametro di forma p ($p \geq 1$), funzione della curtosi, rappresenta una famiglia di curve simmetriche molto utile per la descrizione di errori accidentali.

Esistono molti metodi per la generazione di numeri pseuso casuali da questa famiglia di curve, fondati su trasformazioni di variabili casuali, su metodi di accettazione-rifiuto o su metodi fondati su rapporti di numeri uniformi. In un precedente lavoro abbiamo presentato un metodo fondato su una generalizzazione della nota formula di Box-Muller; nel presente lavoro viene prima presentato un miglioramento di questa routine, che ha il grande pregio di essere codificabile con poche istruzioni; nel seguito del lavoro vengono presentati due nuovi algoritmi fondati su regole di accettazione-rifiuto basati su tecniche di compressione della funzione di densità. Le prestazioni di questi nuovi metodi vengono confrontate con quelle di altri metodi noti in letteratura, sia dal punto di vista della velocità che da quello della bontà *statistica* delle sequenze ottenute. Gli algoritmi fondati su metodi di accettazione rifiuto proposti, sebbene richiedano una codifica piuttosto lunga, sono risultati di gran lunga i più veloci.

Marcello Chiodi: GENERATION OF PSEUDO RANDOM VARIATES FROM A NORMAL DISTRIBUTION OF ORDER P.Pubblicato su Statistica Applicata (Italian Journal of Applied Statistics) 1995, Vol.7, n.4, pagg. 401-416

GENERATION OF PSEUDO RANDOM VARIATES FROM A NORMAL DISTRIBUTION OF ORDER P.

Marcello Chiodi
Istituto di Statistica, Facoltà di Economia e Commercio di Palermo.

Abstract

In simulation studies concerning the behaviour of estimators under departures from normality, the family of *normal distributions of order p* (also known as *exponential power distributions*) with density:

$$f(z)=[2p^{(1/p)}\Gamma(1+1/p)\sigma_p]^{-1}\exp[-|z-\mu|^p/(p\sigma_p^p)], \qquad p>0,\ z\in\Re \quad (1)$$

plays a central role, because it depends on a shape parameter p ($p\geq 1$) which is related to the kurtosis, so that it is very useful in the description of general symmetrical distributed random errors. A wide variety of random number generator exists for this family, based on transformation methods, on acceptance rejection rules and on ratio-of-uniforms methods.

In a previous paper (Chiodi,1986) we proposed a method founded on a generalization of the well known Box-Muller polar transformation method.

In this paper we first present a new quick polar Box-Muller generalization, similar to that of Marsaglia, faster than our previously proposed method. Secondly, in order to gain fastness, in spite of shortness of code, two *squeeze methods* are presented. The two proposed squeeze methods are founded on two decompositions of the range of x, in three and six parts, respectively. They resulted to be faster than other compared routines for several values of p: the six parts algorithm is on average at least twice as fast as other transformation methods. The statistical behaviour of the sequences obtained revealed also to be very satisfactory.

Marcello Chiodi: GENERATION OF PSEUDO RANDOM VARIATES FROM A NORMAL DISTRIBUTION OF ORDER P.Pubblicato su Statistica Applicata (Italian Journal of Applied Statistics) 1995, Vol.7, n.4, pagg. 401-416

1. Introduction

In simulation studies concerning the behaviour of estimators under departures from normality, the family of *normal distribution of order p* (also known as *exponential power distributions*) plays a central role, because it depends on a shape parameter p which is related to the kurtosis (Lunetta,1963; Box & Tiao, 1973; Mineo, 1983,1989, Chiodi, 1994). A wide variety of random number generators exist for this family, based on transformation methods (Chiodi, 1986; Johnson, 1979), on acceptance-rejection rules (Devroye,1986; Tadikamalla,1980) and on ratio-of-uniforms methods (Young et al.,1988, Barabesi 1993a and 1993b). The goodness of a random number generator can be evaluated by different indicators, of statistical as well as computational nature; mainly, a good generator should be simple to code and quick, and should also satisfy a set of goodness of fit tests in order to guarantee its usefulness in the simulation of sampling from a given distribution. In a previous paper (Chiodi, 1986) we proposed a method for the generation of random numbers from a normal distribution of order p (p>1), founded on a generalization of the well known Box-Muller (1958) polar transformation method. Our generator resulted to be quite good from a statistical point and required few lines of code in any scientific programming language, but resulted to be not very fast. Recently, Barabesi (1993a) proposed a refinement of a ratio-of-uniform method (Young et al., 1988), which appears to be quite fast.

First in this paper a faster improvement of our previous routine (Chiodi, 1986), founded on a polar transformation similar to that of Marsaglia (1964), is presented.

Secondly, in order to get faster generator, in spite of shortness of code, *squeeze methods* can be proposed to obtain very fast routines, not very simple to code, but fastest than many other known routines, for any value of p > 1, using some simple device to obtain suitable decomposition of the density function.

The present paper is so organised: in the second section we review some of the properties of the normal distribution of order p; in the third section the principal known methods to generate random number from that distribution are recalled; in the fourth section a new quick polar Box-Muller generalization is presented; in the fifth section two new squeeze methods founded on two decompositions of the x-axis are exposed; in the sixth and last section we report brief results of some simulations to compare the performances of some of the best routines, in terms of execution times and of goodness of fit in the tails and in the center of the distributions.

Marcello Chiodi: GENERATION OF PSEUDO RANDOM VARIATES FROM A NORMAL DISTRIBUTION OF ORDER P.Pubblicato su Statistica Applicata (Italian Journal of Applied Statistics) 1995, Vol.7, n.4, pagg. 401-416

2. The normal distribution of order p.

The family of normal distributions of order p, also known as exponential power distribution, was first introduced by Subbotin (1923) and it has density given by:

$$f(z)=[2p^{(1/p)}\Gamma(1+1/p)\sigma_p]^{-1}\exp[-|z-\mu|^p/(p\sigma_p^p)], \qquad p>0,\ z\in\Re \quad (1)$$

This particular form has been introduced by Vianelli (1963), and it has also been studied by Lunetta (1963) and Mineo (1978,1986). With this parametrization, μ is the true value of a quantity, which observed values z_i are affected by random errors, with dispersion given by σ_p, the mean absolute deviation of order p. For each value of p, a different kind of error distribution is found. It is a class of symmetrical unimodal, bell-shaped (for p>1) curves useful for the description of a wide class of distributions of symmetrical random errors, which includes, as special cases, the Laplace (p=1), the normal (p=2) and the uniform distribution ($p\rightarrow\infty$). Here μ and σ_p are location and scale parameters, respectively, and p is a shape parameter related with the kurtosis, since we have:

$$\mu=E[Z]; \qquad \sigma_p = \{E[|Z-\mu|^p]\}^{1/p} \text{ and}$$
$$\beta_2 = E[|Z-\mu|^4]/\{E[|Z-\mu|^2]\}^2 = \Gamma(1/p)\Gamma(5/p)/\{\Gamma(3/p)\}^2.$$

Letting $X=(Z-\mu)/\sigma_p$, we obtain the standardized form of (1):

$$f(x)=[2p^{(1/p)}\Gamma(1+1/p)]^{-1}\exp[-|x|^p/p], \qquad p>0,\ x\in\Re \qquad (2)$$

Inspection of (2) reveals that the direct inversion method cannot be used, since the probability integral is not expressed in a closed form, so that other methods must be employed.

3. Random number generation from a standardized normal distribution of order p.

In order to generate pseudo random number with density (2), for p>1, different methods can be used. In a previous paper (Chiodi, 1986), we proved that if

$$X=W\cdot[-p\log(U)]^{1/p}\, Y^{1/p}, \qquad (3)$$

where: Prob{W=1} = Prob{W=-1} = 1/2
U is a uniform variate in the range 0-1
Y is a beta variate with parameters α=1/p and β=1-1/p,

then X is distributed as a standardized normal variate of order p. In the above paper we obtained this result as a generalization of the well known Box-Muller (1958) method.

Johnson (1979) used the property that $|Z|^p$ follows a gamma distribution. Tadikamalla (1980) proposed two rejection methods: the first is based on an exponential majorizing function, the second on a normal majorizing function, so that it can be used only for $p \geq 2$; Devroye (1986) used the unimodality of the density; Young (1988) and then Wakefield (1991) proposed efficient methods based on the ratio-of-uniform approach (Kinderman, Monahan, 1977); recently Barabesi (1993a) proposed a refinement of this technique in order to obtain a better theoretical efficiency, that is, a higher probability of acceptance.

4. An improvement of the Box-Muller generalization

Following a reasoning similar to that employed by Marsaglia for his polar transformation method for generating numbers from a standard normal distribution (Marsaglia, Bray, 1964), which avoids the trigonometric functions evaluation in the Box-Muller formula, we can obtain a faster version of our earlier generator (algorithm A, Chiodi,1986) in the following way:

let U and V two uniformly distributed independent variates in the range [-1;+1], and let:

$Z=|U|^p + |V|^{p/(p-1)}$,

then, conditioned on $Z \leq 1$, Z is uniformly distributed, $|U|^p/Z$ has a Beta distribution (Johnk, 1964, Chiodi, 1986) and, if we put:

$X=U\,[-p \log (Z)/Z]^{(1/p)}$,

then X follows a normal distribution of order p.

The coding in any scientific programming language of this algorithm is straightforward and has the advantage that it needs the generation only of two uniformly distributed numbers

5. Two *squeeze* quick generators

If our main interest is to obtain a quick routine, regardless of the compactness and the theoretical properties of a transformation method, it is known that very quick routines for the generation of pseudo random numbers from a distribution with density f(x), can be obtained by means of *squeeze methods* (Marsaglia, 1977, 1984; Schmeiser, Lal., 1980), based on a slight improvement of the general acceptance-rejection rule (Von Neumann, 1951). The key of the method is to find two functions, b(x) and h(x), such that:

$$b(x) \leq f(x) \leq h(x) \quad \text{for all } x,$$

so that the desired density f(x) is *squeezed* between b(x) and h(x).

The method can be so sketched:

(a) Generate X from a distribution of density r(x), with $r(x)=h(x) / \int_{-\infty}^{+\infty} h(x)dx$;

(b) generate V from a uniform distribution on [0-1];

(c) if $V \le b(X)/h(X)$, deliver X as a number from f(x) else

(d) if $V \le f(X)/h(X)$, deliver X as a number from f(x); otherwise reject X and go to step (a)

The functions b(x) and h(x) should be simple to compute and the generation from r(x) should be simple and quick. Furthermore, the best results are obtained when the "quick acceptance test", that is, $b(X)/h(X)>V$, has an high probability of being passed: it happens when both b(x) and h(x) are close to f(x).

We experimented different choices of b(x) and h(x). Here we sketch briefly only the two choices which performed best in terms of execution times and in terms of fitness to the theoretical distribution.

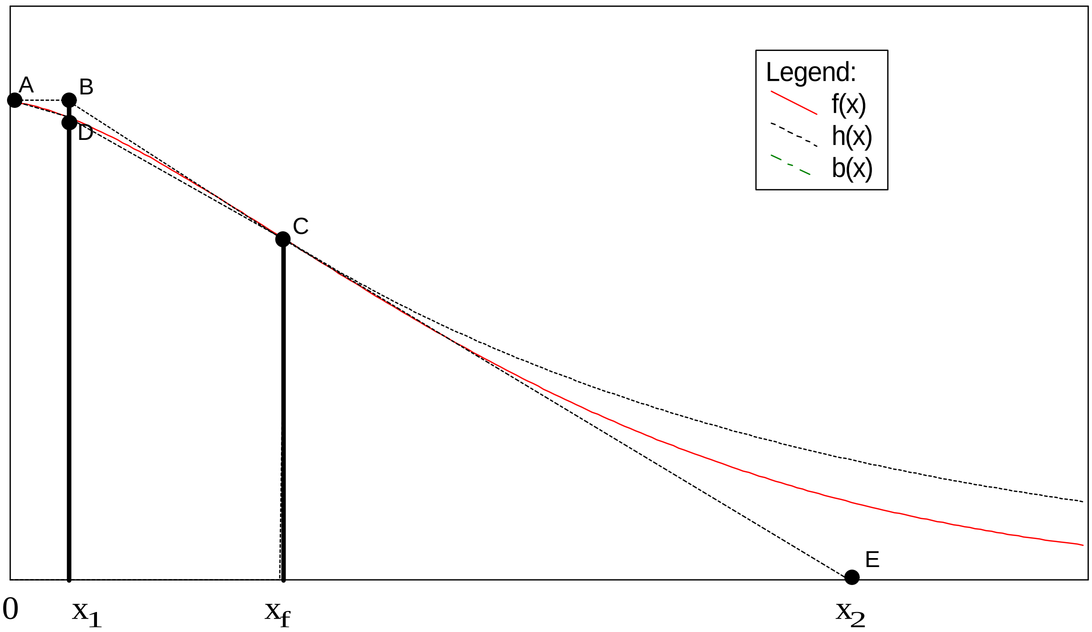

Fig.1 Three-part decomposition (algorithm SQ3, figure obtained for p=1.5)

The first proposed method (SQ3) is founded on a decomposition of the positive x-axis in three parts, as described in fig.1. The key of this method is t(x), the straight line tangent to f(x) in its point of inflection, C. Let:

$x_f=(p-1)^{1/p}$, so that $f''(x_f)=0$,

x_1 such that $t(x_1)=f(0)$;
x_2 such that $t(x_2)=0$
and define the points A,B,C,D and E and their co-ordinates in this way:
$A(0;f(0))$; $B(x_1;f(0))$; $C(x_f;f(x_f))$; $D(x_1;f(x_1))$; $E(x_2;0)$.

So h(x) is chosen to be uniform between 0 and x_1 where: $h(x)=y_0$; triangular in x_1-x_f, where $h(x)=y_f+(x-x_f)(y_0-y_f)/(x_0-x_f)$; for the tail ($x>x_f$) the majorizing curve is given by an exponential function:

$h(x) = k_1 \exp(- k_2 x)$ (see also Schmeiser, 1980),

where k_1 and k_2 are chosen by imposing the constrain that h(x) is tangent to f(x) in x_f.

For this algorithm b(x) is the piecewise line ADCE, for $x<x_2$.

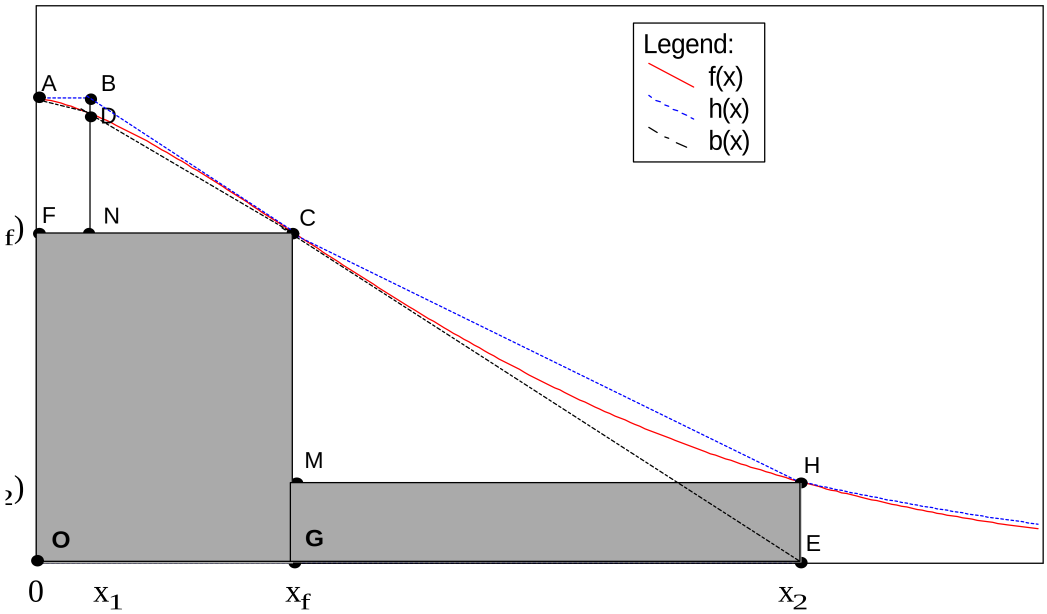

Fig.2 Six-areas decomposition (algorithm SQ6, figure obtained for p=1.5)

The second proposed method (SQ6) is based on a more complex decomposition of the area under f(x) (for x>0) in six parts, as described in fig.2. The key of this method is again the straight line t(x). With respect to the first decomposition we have some more points: let

$F(0;f(x_f))$, $G(x_f;0)$, $H(x_2;f(x_2))$, $M(x_f,f(x_2))$, $N(x_1,f(x_f))$ and $O(0,0)$.

So h(x) is chosen to be uniform between 0 and x_1, triangular in x_1-x_f, again triangular in x_f-x_2, and for the tail ($x>x_2$) the majorizing curve is given by:

$h(x) = y_0 (x/x_2)^{p-1} \exp(-x^p/p)$, which is an extension of the rule proposed by Marsaglia to generate variates from the tails of the normal distribution, while b(x) is the same of the previous algorithm.

Following the decomposition shown in fig. 2, where the whole area under h(x) is divided in six parts, the aim of the technique is the following (for the sake of brevity, the whole algorithm is exposed in the appendix):

The main steps of the algorithm are:

- generate a uniform variate U and accept it as the desired variates, after a suitable linear transformation, if it is less then the normalized areas of the two rectangles
- otherwise, according to the the value of U, compute X by inversion of the proper uniform distribution (rectangle ABNF), or of a triangular ditribution (triangles BCN and CHM), or of the tail majorizing curve given by (4) (area bounded by h(x) to the right of the point E) and then generate V from a uniform distribution and if b(X)/h(X)>V (pre-test), accept X; else if f(X)/h(X)>V then accept X after a random assignment of the sign, else reject X and begin again with step (1). Details on the algorithm are given in the appendix.

As usual for acceptance-rejection methods, the theoretical efficiency E_p is defined (Rubinstein, 1985, p.46) as:

$$E_p=\text{Prob}\{V<f(X)/h(X)\}.$$

Values of E_p for the two proposed squeeze algorithms for 16 selected values of p are reported in Tab. I. For the sake of brevity, we avoid to report formulae needed for their computation.

p	**1.01**	**1.10**	**1.25**	**1.50**	**1.75**	**2.00**	**2.25**	**2.50**
Efficiency SQ3	.9608	.8460	.8046	.8091	.8286	.8475	.8633	.8762
Efficiency SQ6	.9462	.9344	.9362	.9457	.9523	.9560	.9582	.9597
p	**2.75**	**3.00**	**4.00**	**5.00**	**6.00**	**8.00**	**10.00**	**20.00**
Efficiency SQ3	.8868	.8956	.9195	.9340	.9438	.9564	.9642	.9812
Efficiency SQ6	.9608	.9618	.9651	.9682	.9711	.9756	.9791	.9880

Tab. I Theoretical efficiencies of the two proposed squeeze algorithms corresponding to 16 values of p

As it is clear from Fig. 2, the aim of algorithm SQ6 is to squeeze efficiently the function f(x), so that we can expect very fast and efficient generations for any value of p, since the function b(x) is also close to f(x). The generation from h(x), given this decomposition, can be very fast also thanks to the presence the two rectangular areas for which only one uniform variate is needed.

6. Comparison of the performances of the proposed algorithms.

We coded the proposed algorithms in FORTRAN (MS FORTRAN Compiler 5.1 on a personal computer with a 80486 processor) and compared them with the two Box Muller generalizations, with the Barabesi algorithm and the two rejection methods of Tadikamalla, because the last ones resulted to be the best generators (in term of speed) for different ranges of values of p, $p>1$ and for $p<2$, respectively, according to the simulations reported by Barabesi (1993a) (actually the comparisons between algorithms have been performed on 16 values of p; in this paper we preferred to report only 6 values of p, to avoid large tables).

The comparison is made for the case of subsequent calls of the subroutines with p fixed, so that the initialization time for each generation routine is spent only at the first call, which is the most general case in the applications. In Tab. II we report the average times in μsec. computed on 500,000 numbers generated for each value of p and for each of the seven compared methods, where:

EL is the Barabesi proposal,

EC is our first proposed routine (Chiodi, 1986),

EC2 is the new proposed polar transformation exposed in section 4,

SQ3 is the three-parts squeeze method,

SQ6 is the six-areas squeeze method and

ED and EN are the two Tadikamalla methods (where EN is valid only for $p \geq 2$).

To obtain the average times reported, we subtracted the amount of time needed in FORTRAN language for the execution of empty DO cycles from the total execution times. We employed well tested multiplicative congruential generators (Downham, Roberts, 1967; Golder, Settle, 1976) to obtain uniformly distributed numbers. In Fig. 3, the same values are reported, for $p<4$.

	ALGORITHMS						
P	EL	EC	EC2	SQ3	SQ6	ED	EN
1.01	172.38	176.02	161.98	105.50	80.56	110.44	- - -
1.25	162.96	192.06	178.12	114.30	69.26	120.22	- - -
1.50	155.80	198.98	185.36	104.62	61.88	129.00	- - -
2.50	140.00	201.08	187.44	79.68	53.32	154.18	136.70
4.00	130.44	195.14	181.30	67.50	51.44	178.54	157.46
10.00	120.12	184.16	169.76	57.94	50.14	220.96	194.04

Tab. II Average execution time (in msec) evaluated on 500.000 number generations for each of the compared algorithms and for 6 selected values of p.

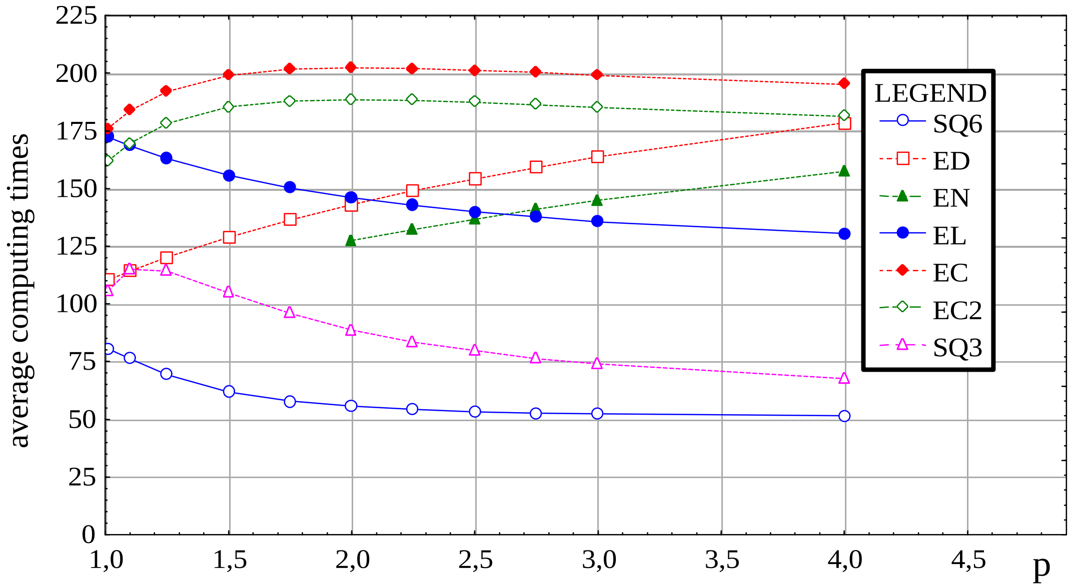

Fig.3 Plot of computing times of the seven routines vs. p evaluated on 11 values of p<4

As can be seen from Tab. II, the SQ3 and the SQ6 methods are the fastest, but the SQ6 method is faster for any value of p, so we can state that the SQ6 method has the best overall performances.

About our previous slow routine (EC), we must observe that even if slower than EL routine, it is not so slower as claimed in the Tab. IV of Barabesi (1993a). Furthermore, the improvement proposed in section 4 of the present paper (routine EC2) provide a faster algorithm with respect to the similar previous routine (EC).

p		ALGORITHMS						
		EL	EC	EC2	SQ3	SQ6	ED	EN
1.01	**total**	**42.06**	***56.01**	**42.27**	**45.84**	**52.05**	**39.24**	**- - -**
	tails	*17.56*	*10.80*	*10.04*	*8.07*	***25.42*	*7.46*	- - -
	center	*6.71*	*14.00*	*15.54*	*15.30*	*4.18*	*11.07*	- - -
1.25	**total**	**48.01**	**46.24**	**43.14**	**44.34**	**29.73**	**34.57**	**- - -**
	tails	*8.94*	*11.57*	*12.83*	*11.77*	*11.71*	*5.55*	- - -
	center	*12.26*	*13.03*	*8.47*	*5.28*	*8.22*	*5.13*	- - -
1.50	**total**	**37.13**	**44.21**	**33.61**	**47.14**	**34.05**	****64.68**	**- - -**
	tails	*12.39*	*9.75*	*11.30*	*14.03*	*9.1*	***23.72*	- - -
	center	*7.54*	*4.45*	*6.67*	*8.19*	*3.58*	*11.41*	- - -
2.50	**total**	**30.83**	**50.24**	**33.47**	**32.06**	**41.66**	**43.73**	**40.74**
	tails	*4.93*	*7.50*	*8.95*	*8.20*	*12.11*	*13.49*	*8.34*
	center	*5.94*	*6.99*	*7.93*	*7.12*	*7.17*	*10.42*	*5.35*
4.00	**total**	**29.32**	**27.54**	**30.87**	**43.20**	**33.85**	**37.96**	**42.86**
	tails	*15.36*	*3.38*	*4.90*	*9.37*	*7.87*	*9.58*	**19.80*
	center	*5.20*	*8.04*	*7.57*	*9.83*	*7.59*	*6.90*	*5.96*
10.00	**total**	**43.48**	**49.95**	**30.87**	**44.14**	**33.94**	**38.69**	**51.51**
	tails	*5.7*	*13.90*	*6.86*	*11.65*	*9.59*	*7.62*	*9.01*
	center	*7.1*	*13.26*	*4.02*	*8.77*	*6.78*	*6.29*	*13.27*

Tab III. Pearson's X^2 values of goodness of fit test evaluated for each of the algorithms and for six values of p on the whole distributions (40 classes), on the tails (10 class) and on the center of the distributions; (*) stands values beyond the 5% critical c^2 values, (**) for 1% critical values.

In Tab. III the Pearson X^2 values of Tab III, evaluated on 40 classes for each sequence are reported, together with the X^2 values for the tails and for the center of the distributions; each of these partial X^2 values is computed on ten classes. For each values of p the classes have been computed in order to have a theoretical probability of 0.001 for the two extreme classes; the other 38 internal classes have equal width. The behaviour of the generators is generally satisfactory from the statistical point of view of the goodness of fit to the theoretical distributions, as we can see from the values of Tab.III. There are only two significant values at the 1% level; however we performed the same simulation experiment again, obtaining non significant values. As usual, significant values should be judged very carefully, because many tests are performed on the same generators, so that some significant value can be obtained also with good routines.

However, if the aim of the comparison is merely fastness, our squeeze generators, or other similar that can be implemented, should be used.

NOTES

This research has been supported by M.U.R.S.T. funds (60%) in the framework of the research project: "Sulla distribuzione di campionamento dei test k_p, t_p, F_p, χ^p".

REFERENCES

Barabesi, L., 1993, *Optimized ratio-of-uniform method for generating exponential power variates*, Statistica Applicata, 5, 2, 149-155.

Barabesi, L., 1993, *Random variate generation by using ratio-of-uniform method*, Univ. degli studi di Siena, Dip. di Metodi quantitativi, Collana di Pubblicazioni, Serie ricerca monografie,1.

Box, G . E. P. and Muller, M. E., 1958, *A note on the generation of random normal deviates*, Annals of Mathematical Statistics, 29, 610-611.

Box, G. E. P. and Tiao, G.C., 1973, *Bayesian inference in statistical analysis*, Addison-Wesley, Reading, Mass, 1-588.

Chiodi, M., 1986, *Procedures for generating pseudo-random numbers from a normal distribution of order p (p>1)*, Statistica Applicata, 1, 7-26.

Chiodi, M., 1994, *Approssimazioni saddlepoint alle distribuzioni campionarie degli stimatori di massima verosimiglianza dei parametri delle curve normali di ordine p per piccoli campioni.* Atti della XXXVII Riunione Scientifica della SIS Ed. C.I.S.U. Roma; 1, 139-146.

Devroye, L., 1986, *Non-uniform random variate generation*, Springer-Verlag Ed., New York.

Downham, D.Y., Roberts, F.D.K., 1967, *Multiplicative congruential pseudo-random number generators.* Computer Journal, 10, 74-77.

Golder, E.R., Settle, J.G., 1976, *The Box-Muller method for generating pseudo-random normal deviates.* Applied Statistics, 25, 12-20.

Johnson, M. E., 1979, *Computer generation of the exponential power distributions*, Journal of Statistical Computation and simulation, 9, 239-240.

Kinderman, A. J. and Monahan, J. F., 1977, *Computer generation of random variable using the ratio of uniform deviates*, ACM Transaction on Mathematical Software, 3, 257-260.

Lunetta, G., 1963, *Di una generalizzazione dello schema della curva normale*. Annali della Facoltà di Economia e Commercio di Palermo, 17, 2, 235-244.

Marsaglia, G., Bray, T. A., 1964, *A convenient method for generating normal variables.* SIAM rev. 6, 260-264.

Marsaglia, G., 1977, *The squeeze method for generating gamma variates*, Journal of the American Statistical Association, 79, 218-221.

Marsaglia, G., 1984, *The exact-approximation method for generating random variables in a computer,* Journal of the American Statistical Association, 79, 218-221.

Mineo, A., 1978, *La stima dei parametri delle curve normali di ordine r per intervalli finiti e di quelle per intervalli infiniti*, Studi in onore di Paolo Fortunati, C.L.E.U.B. ed., Bologna, 1, 565-577.

Mineo, A., 1983, *La miglior combinazione delle osservazioni per la stima dei parametri di intensità e di dispersione*, Atti del Convegno della S.I.S., Trieste, 1, 463-487.

Mineo, A., 1989, *The norm-p estimation of location, scale and simple linear regression parameters*, Lecture notes in Statistics, Statistical Modeling Proceedings, Trento, 222-223.

Rubinstein, R. Y., 1981, *Simulation and the Monte Carlo Method,* Wiley ed.

Schmeiser, B. W., 1980, *Generation of variates from distribution tails,* Operation Research, 28,4, 1012-1017.

Schmeiser, B. W., Lal, R., 1980, *Squeeze methods for generating gamma variables,* Journal of the American Statistical Association, 75, 679-682.

Subbotin, M. T., 1923, *On the law of frequency of errors*, Matematicheskii Sbornik, 31, 296-301.

Tadikamalla, P. R., 1980, *Random sampling from the exponential power distribution*, Journal of the American Statistical Association, 75, 683-686.

Vianelli, S.,1963, *La misura della variabilità condizionata in uno schema generale delle curve normali di frequenza.* Statistica, 33. 447-474.

Von Neumann, J., 1951, *Various techniques used in connection with random digits*, U.S: Nat. Bur. Stand. Appl. Math. Ser., N. 12, 36-38.

Wakefield, J. C.; Gelfand, A. E. and Smith, A. F. M., 1991, *Efficent generation of random variates via the ratio-of-uniform method*, Statistics and Computing, 1, 129-133.

Young, D. M., Turner, D. W. and Seaman, J. W., 1988, *A ratio-of-uniforms method for generating exponential power variates*, Computer Science and Statistics, Proceedings of the 20th Symposium on the Interface, Wegman (ed.), 627-629.

APPENDIX

ALGORITHM SQ6

<u>Inizialization steps:</u>

Consider the points of fig. 2. and denote by $A(P_1P_2...P_k)$ the area of a polygon with vertexes: $P_1P_2...P_k$ and let:

$$S_h = \int_{x_2}^{+\infty} h(x)dx = f(0)\exp(-x_2^p/p)/x_2^{p-1} \quad \text{(area under h(X) for } X > x_2\text{)}$$

$A_1 = A(OGCF)$; $A_2 = A_1 + A(FNBA)$; $A_3 = A_2 + A(BNC)$; $A_4 = A_3 + A(GEHM)$; $A_5 = A_4 + A(CMH)$; $A_6 = A_5 + S_h$ and compute the values of the A_i with the following relationships:

$A_1 = x_f\, f(x_f)$; $A_2 = A_1 + x_1(f(x_0)-f(x_f))$; $A_3 = A_2 + (x_f - x_1)(f(x_0)-f(x_f))/2$;

$A_4 = A_3 + (x_2 - x_f)f(x_2)$; $A_5 = A_4 + (x_2 - x_f)(f(x_f)-f(x_2))/2$;

$A_6 = A_5 + f(x_0)\exp(-x_2^p/p)/(x_2^{(p-1)})$

Now, after these inizialization steps for a fixed p, the main steps of the algorithm are:

(1) generate a uniform variate U_0; *if* $U_0 < 0.5$ *then* let $U = 2U_0A_6$ and $W=-1$ *else* let $U = 2(1-U_0)A_6$ and $W=+1$(random assignment of the sign) and *then* select the proper rule according to the value of U:

(2) *if* $U \le A_1$ (first area: the random point is inside rectangle OGCF), let $X=U/f(x_f)$ and go to step 8;

(3) *if* $A_1 < U \le A_2$, (second area, rectangle FNBA), compute X by inversion of the proper uniform distribution: $X=(U-A_1)/(f(x_0)-f(x_f))$ and generate V from a uniform distribution and *then*:

if b(X)/h(X)>V (pre-test),*then* go to step 8;

else *if* f(X)/h(X)>V *then* go step 8

else reject the number and begin again with step 1

(4) *if* $A_2 < U \le A_3$ (third area, triangle BNC), let $U_1=(U - A_2)/(A_3 - A_2)$ and generate another uniform number U_2, so that $U_3=\min(U_1,U_2)$ is a triangular distributed variate; compute X by a linear transformation: $X=x_1+U_3(x_f-x_1)$; then generate V from a uniform distribution and

if b(X)/h(X)>V (pre-test), *then* go to step 8

else if f(X)/h(X)>V *then* go to step 8 *else* reject X and begin again with step 1

(5) *if* $A_3 < U \le A_4$ (fourth area: rectangle GEHM), let $X=(U-A_3)/f(x_2)$ and go to step 8

(6) *if* $A_4 < U \le A_5$ (third area, triangle BNC), let $U_1=(U - A_4)/(A_5 - A_4)$ and generate another uniform number U_2 , so that $U_3=\min(U_1,U_2)$ is a triangular distributed variate; compute X by a linear transformation: $X=x_f+U_3(x_2-x_f)$; *then* generate V from a uniform distribution and

if b(X)/h(X)>V (pre-test), *then* go to step 8

else if f(X)/h(X)>V *then* go to step 8 *else* reject X and begin again with step 1

(7) *if* $U>A_5$ (sixth area, tail of the distribution to the right of x_2),

let $X=[x_2^p - p \log((U - A_5) / (A_6-A_5)]^{1/p}$ *then* generate V from a uniform distribution and *if* $V<(x_2 /X)^{p-1}$ *then* go to step 1 *else* reject X and begin again with step 1

(8) let $X_p=W\cdot X$ and accept X_p as the desired pseudo random number from a normal distribution of order p